(i) Printed Pages : 3 Roll No. ....

(ii) Questions : 8 **Sub. Code : 0 2 3 8**

Exam. Code : 0 0 3

B.A./B.Sc. (General) 3rd Semester

1125

MATHEMATICS

Paper : I : Advanced Calculus–I

Time Allowed : Three Hours] [Maximum Marks : 30

Note :– Attempt **five** questions in all, selecting at least **two** from each Unit. All questions carry equal marks.

UNIT–I

1. (a) Let a function f be defined by

$$f(x, y) = \frac{x^3 - y^3}{x^3 + y^3}, \ (x, y) \neq (0, 0).$$

Show that the two iterated limits $\lim_{x \to 0} \left\{ \lim_{y \to 0} f(x, y) \right\}$ and $\lim_{y \to 0} \left\{ \lim_{x \to 0} f(x, y) \right\}$ exist by the simultaneous limit $\lim_{(x, y) \to (0, 0)} f(x, y)$ does not exist.

(b) Discuss the continuity of the function

$$f(x, y) = \begin{cases} x\, y \sin \dfrac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

at the point (0, 0).

2\. (a) If $x^x y^y z^z = c$, show that $\dfrac{\partial^2 z}{\partial x \partial y} = -(x \log e x)^{-1}$, when $x = y = z$.

(b) If $u = f(r)$, where $r = \sqrt{x^2 + y^2}$, prove that :

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r} f'(r).$$

3\. (a) Show that $f(x, y) = \sin x + \cos y$ is differentiable at every point of $\mathbf{R}^2$.

(b) Show that $f_{xy}(0, 0) \neq f_{yx}(0, 0)$, where

$$f(x, y) = \begin{cases} x^2 \tan^{-1}\left(\dfrac{y}{x}\right) - y^2 \tan^{-1}\left(\dfrac{x}{y}\right) & \text{if } x\, y \neq 0 \\ 0 & \text{if } x\, y = 0 \end{cases}$$

4\. (a) For the function $f = \dfrac{y}{x^2 + y^2}$, find the value of directional derivative making an angle 30° with the positive x-axis at the point (0, 1).

(b) Prove that curl curl $\vec{A}$ = grad div $\vec{A} - \nabla^2 \vec{A}$.

UNIT–II

5\. (a) State and prove Euler's theorem on homogeneous functions of two variables.

(b) State Taylor's theorem for function of two variables and use it to expand $x^2y - 3y + 3$ in powers of $x + 1$ and $y - 2$.

6. (a) If u, v, w are the roots of the equation

$$(\lambda - x)^3 + (\lambda - y)^3 + (\lambda - z)^3 = 0$$

prove that :

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = \frac{2(x - y)(y - z)(z - x)}{(u - v)(v - w)(w - u)}.$$

(b) Show that the functions

$$u = x + y - z,\ v = x - y + z,\ w = x^2 + y^2 + z^2 - 2yz$$

are not independent of one another. Also find the relation between u, v and w.

7. (a) Find the extreme values of the function :

$$f(x, y) = (x - y)^4 + (y - 1)^4.$$

(b) Show that the volume of the largest parallelopiped that can be inscribed in the sphere

$$x^2 + y^2 + z^2 = a^2 \text{ is } \frac{8a^2}{3\sqrt{3}}.$$

8. (a) Prove that the envelope of the circles which pass through the centre of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and have their centres upon its circumference is the curve $x^2 + y^2 = 4(a^2x^2 + b^2y^2)$.

(b) If ρ_1 and ρ_2 are the radii of curvature at the corresponding points of a cycloid and its evolute, prove that :

$$\rho_1^2 + \rho_2^2 = \text{constant}.$$