Exam.Code:0003
Sub. Code: 0241

2121

B.A./B.Sc. (General) Third Semester

Mathematics

Paper – A: Advanced Calculus – I

Time allowed: 3 Hours

Max. Marks: 30

NOTE: Attempt five questions in all, selecting atleast two questions each Unit. All questions carry equal marks.

x-x-x

UNIT - I

1. (a) Let $f: \mathbf{R}^2 \to \mathbf{R}$ be defined by $f(x, y) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational.} \end{cases}$

Show that $\lim_{(x,y)\to(a,b)} f(x, y)$ for any point (a, b) does not exist.

(b) Examine the function $f(x, y) = \dfrac{xy^3}{x^2+y^2}$, $(x, y) \neq (0, 0)$ and $f(0, 0) = 0$ for continuity at $(0, 0)$.

2. (a) If $u = \log(x^3 + y^3 + z^3 - 3xyz)$, show that $\left(\dfrac{\partial}{\partial x}+\dfrac{\partial}{\partial y}+\dfrac{\partial}{\partial z}\right)^2 u = -\dfrac{9}{(x+y+z)^2}$.

(b) If V is a function of two variables x and y and $x = r\cos\theta$, $y = r\sin\theta$, prove that

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}=\frac{\partial^2 V}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2}+\frac{1}{r}\frac{\partial V}{\partial r}.$$

3. (a) Show that $f(x, y) = \sin x + \cos y$ is differentiable at every point of $\mathbf{R}^2$.

(b) Show that $f_{xy}(0, 0) \neq f_{yx}(0, 0)$, where

$$f(x,y) = \begin{cases} x^2 \tan^{-1}\left(\dfrac{y}{x}\right) - y^2 \tan^{-1}\left(\dfrac{x}{y}\right) & \text{if } xy \neq 0 \\ 0 & \text{if } xy = 0. \end{cases}$$

4. (a) Find the directional derivative of $\phi = x^2y^2z^2$ at the point P(1, 1, – 1) in the direction of tangent to the curve $x = e^t$, $y = 2\sin t + 1$, $z = t - \cos t$ at $t = 0$.

(b) Show that $\nabla.(\vec{f} \times \vec{g}) = \vec{g}.(\nabla \times \vec{f}) - \vec{f}.(\nabla \times \vec{g})$.

UNIT – II

5. (a) State and prove Euler's theorem on homogeneous functions of two variables.

(b) Expand $\tan^{-1}\dfrac{y}{x}$ in the neighbourhood of the point (1, 1).

6. (a) If u, v, w are the roots of the equation $(\lambda - x)^3 + (\lambda - y)^3 + (\lambda - z)^3 = 0$, prove that

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = -\frac{2(x-y)(y-z)(z-x)}{(u-v)(v-w)(w-u)}.$$

(b) Show that the functions $u = x + y - z$, $v = x - y + z$, $w = x^2 + y^2 + z^2 - 2yz$ are not independent of one another. Also find the relation between u, v and w.

P.T.O.

7. (*a*) Find the extreme values of the function $f(x, y) = 12x^2 + 8xy + y^2 + 8x^3$.

(*b*) Find the shortest distance from the origin to the hyperbola $x^2 + 8xy + 7y^2 = 225, z = 0$.

8. (*a*) Find the envelope of the family of the curves $\frac{a^2}{x}\cos\theta - \frac{b^2}{y}\sin\theta = c$ for different values of θ.

(*b*) If ρ_1 and ρ_2 are the radii of curvature at the corresponding points of a cycloid and its evolute, prove that $\rho_1^2 + \rho_2^2$ = constant.

x-x-x